10.6 Polar Equations of Conics and Kepler's Laws

- **Analyze and write polar equations of conics.**
- **Understand and use Kepler's Laws of planetary motion.**

> **Exploration**
>
> ***Graphing Conics*** Set a graphing utility to *polar* mode and enter polar equations of the form
>
> $$r = \frac{a}{1 \pm b \cos \theta}$$
>
> or
>
> $$r = \frac{a}{1 \pm b \sin \theta}.$$
>
> As long as $a \neq 0$, the graph should be a conic. What values of a and b produce parabolas? What values produce ellipses? What values produce hyperbolas?

Polar Equations of Conics

In this chapter, you have seen that the rectangular equations of ellipses and hyperbolas take simple forms when the origin lies at their *centers*. As it happens, there are many important applications of conics in which it is more convenient to use one of the foci as the reference point (the origin) for the coordinate system. For example, the sun lies at a focus of Earth's orbit. Similarly, the light source of a parabolic reflector lies at its focus. In this section, you will see that polar equations of conics take simpler forms when one of the foci lies at the pole.

The next theorem uses the concept of *eccentricity*, as defined in Section 10.1, to classify the three basic types of conics.

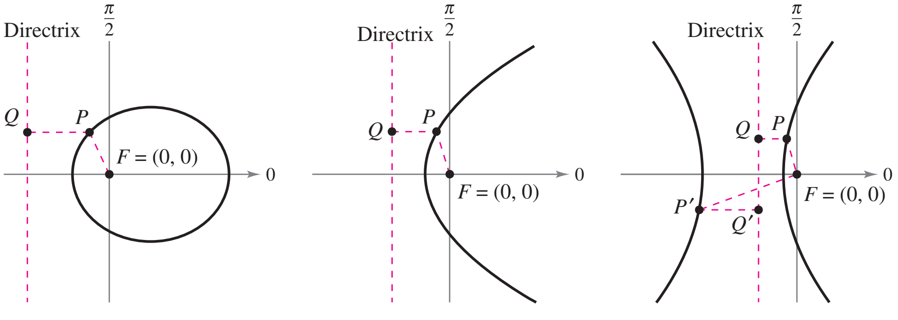
THEOREM 10.16 Classification of Conics by Eccentricity

Let F be a fixed point (*focus*) and let D be a fixed line (*directrix*) in the plane. Let P be another point in the plane and let e (*eccentricity*) be the ratio of the distance between P and F to the distance between P and D. The collection of all points P with a given eccentricity is a conic.

1. The conic is an ellipse for $0 < e < 1$.
2. The conic is a parabola for $e = 1$.
3. The conic is a hyperbola for $e > 1$.

A proof of this theorem is given in Appendix A.
See LarsonCalculus.com for Bruce Edwards's video of this proof.

In Figure 10.57, note that for each type of conic, the pole corresponds to the fixed point (focus) given in the definition.



Ellipse: $0 < e < 1$ $\qquad$ Parabola: $e = 1$ $\qquad$ Hyperbola: $e > 1$

$\frac{PF}{PQ} < 1$ $\qquad$ $PF = PQ$ $\qquad$ $\frac{PF}{PQ} = \frac{P'F}{P'Q'} > 1$

Figure 10.57

The benefit of locating a focus of a conic at the pole is that the equation of the conic becomes simpler, as seen in the proof of the next theorem.

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THEOREM 10.17 Polar Equations of Conics

The graph of a polar equation of the form

$$r = \frac{ed}{1 \pm e \cos \theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e \sin \theta}$$

is a conic, where $e > 0$ is the eccentricity and $|d|$ is the distance between the focus at the pole and its corresponding directrix.

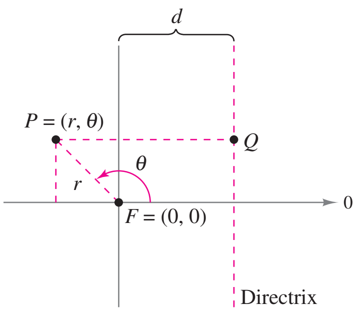


Figure 10.58

Proof This is a proof for $r = ed/(1 + e \cos \theta)$ with $d > 0$. In Figure 10.58, consider a vertical directrix d units to the right of the focus $F = (0, 0)$. If $P = (r, \theta)$ is a point on the graph of $r = ed/(1 + e \cos \theta)$, then the distance between P and the directrix can be shown to be

$$PQ = |d - x| = |d - r \cos \theta| = \left| \frac{r(1 + e \cos \theta)}{e} - r \cos \theta \right| = \left| \frac{r}{e} \right|.$$

Because the distance between P and the pole is simply $PF = |r|$, the ratio of PF to PQ is

$$\frac{PF}{PQ} = \frac{|r|}{|r/e|} = |e| = e$$

and, by Theorem 10.16, the graph of the equation must be a conic. The proofs of the other cases are similar.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The four types of equations indicated in Theorem 10.17 can be classified as follows, where $d > 0$.

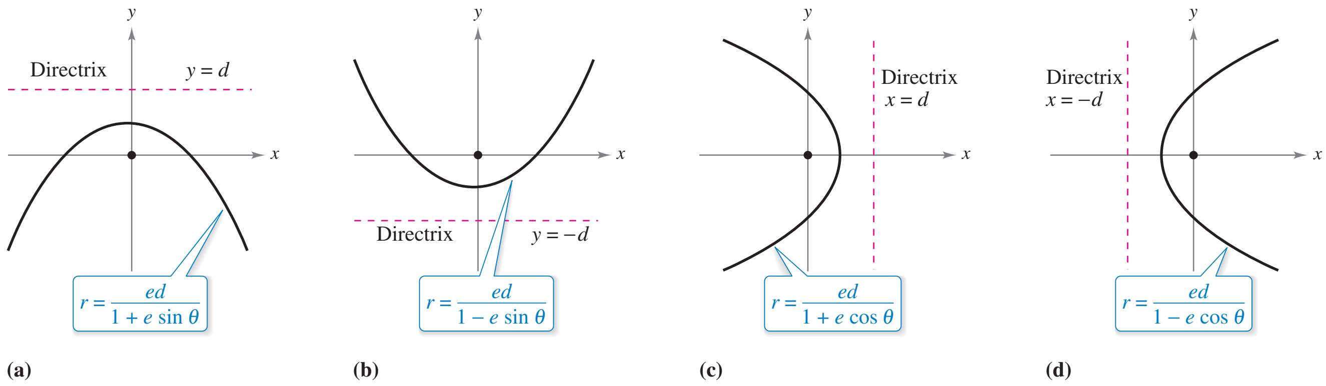
a. Horizontal directrix above the pole: $r = \dfrac{ed}{1 + e \sin \theta}$

b. Horizontal directrix below the pole: $r = \dfrac{ed}{1 - e \sin \theta}$

c. Vertical directrix to the right of the pole: $r = \dfrac{ed}{1 + e \cos \theta}$

d. Vertical directrix to the left of the pole: $r = \dfrac{ed}{1 - e \cos \theta}$

Figure 10.59 illustrates these four possibilities for a parabola. Note that for convenience, the equation for the directrix is shown in rectangular form.



The four types of polar equations for a parabola
Figure 10.59

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EXAMPLE 1 Determining a Conic from Its Equation

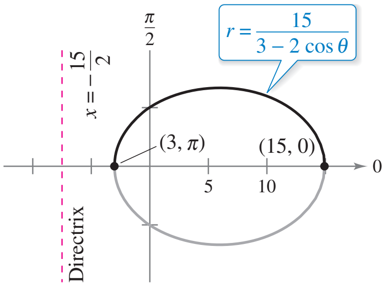
Sketch the graph of the conic $r = \dfrac{15}{3 - 2\cos\theta}$.

Solution To determine the type of conic, rewrite the equation as

$$r = \frac{15}{3 - 2\cos\theta} \qquad \text{Write original equation.}$$

$$= \frac{5}{1 - (2/3)\cos\theta}. \qquad \text{Divide numerator and denominator by 3.}$$

So, the graph is an ellipse with $e = \frac{2}{3}$. You can sketch the upper half of the ellipse by plotting points from $\theta = 0$ to $\theta = \pi$, as shown in Figure 10.60. Then, using symmetry with respect to the polar axis, you can sketch the lower half.



The graph of the conic is an ellipse with $e = \frac{2}{3}$.
Figure 10.60

For the ellipse in Figure 10.60, the major axis is horizontal and the vertices lie at (15, 0) and $(3, \pi)$. So, the length of the *major* axis is $2a = 18$. To find the length of the minor axis, you can use the equations $e = c/a$ and $b^2 = a^2 - c^2$ to conclude that

$$b^2 = a^2 - c^2 = a^2 - (ea)^2 = a^2(1 - e^2). \qquad \text{Ellipse}$$

Because $e = \frac{2}{3}$, you have

$$b^2 = 9^2\left[1 - \left(\tfrac{2}{3}\right)^2\right] = 45$$

which implies that $b = \sqrt{45} = 3\sqrt{5}$. So, the length of the minor axis is $2b = 6\sqrt{5}$. A similar analysis for hyperbolas yields

$$b^2 = c^2 - a^2 = (ea)^2 - a^2 = a^2(e^2 - 1). \qquad \text{Hyperbola}$$

EXAMPLE 2 Sketching a Conic from Its Polar Equation

See LarsonCalculus.com for an interactive version of this type of example.

Sketch the graph of the polar equation $r = \dfrac{32}{3 + 5\sin\theta}$.

Solution Dividing the numerator and denominator by 3 produces

$$r = \frac{32/3}{1 + (5/3)\sin\theta}.$$

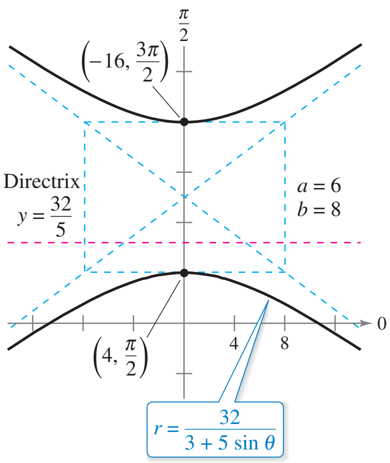
Because $e = \frac{5}{3} > 1$, the graph is a hyperbola. Because $d = \frac{32}{5}$, the directrix is the line $y = \frac{32}{5}$. The transverse axis of the hyperbola lies on the line $\theta = \pi/2$, and the vertices occur at

$$(r, \theta) = \left(4, \frac{\pi}{2}\right) \quad \text{and} \quad (r, \theta) = \left(-16, \frac{3\pi}{2}\right).$$

Because the length of the transverse axis is 12, you can see that $a = 6$. To find b, write

$$b^2 = a^2(e^2 - 1) = 6^2\left[\left(\frac{5}{3}\right)^2 - 1\right] = 64.$$

Therefore, $b = 8$. Finally, you can use a and b to determine the asymptotes of the hyperbola and obtain the sketch shown in Figure 10.61.



The graph of the conic is a hyperbola with $e = \frac{5}{3}$.
Figure 10.61

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JOHANNES KEPLER (1571–1630)

Kepler formulated his three laws from the extensive data recorded by Danish astronomer Tycho Brahe, and from direct observation of the orbit of Mars. *See LarsonCalculus.com to read more of this biography.*

Kepler's Laws

Kepler's Laws, named after the German astronomer Johannes Kepler, can be used to describe the orbits of the planets about the sun.

1. Each planet moves in an elliptical orbit with the sun as a focus.
2. A ray from the sun to the planet sweeps out equal areas of the ellipse in equal times.
3. The square of the period is proportional to the cube of the mean distance between the planet and the sun.*

Although Kepler derived these laws empirically, they were later validated by Newton. In fact, Newton was able to show that each law can be deduced from a set of universal laws of motion and gravitation that govern the movement of all heavenly bodies, including comets and satellites. This is shown in the next example, involving the comet named after the English mathematician and physicist Edmund Halley (1656–1742).

EXAMPLE 3 **Halley's Comet**

Halley's comet has an elliptical orbit with the sun at one focus and has an eccentricity of $e \approx 0.967$. The length of the major axis of the orbit is approximately 35.88 astronomical units (AU). (An astronomical unit is defined as the mean distance between Earth and the sun, 93 million miles.) Find a polar equation for the orbit. How close does Halley's comet come to the sun?

Solution Using a vertical axis, you can choose an equation of the form

$$r = \frac{ed}{(1 + e \sin \theta)}.$$

Because the vertices of the ellipse occur when $\theta = \pi/2$ and $\theta = 3\pi/2$, you can determine the length of the major axis to be the sum of the r-values of the vertices, as shown in Figure 10.62. That is,

$$2a = \frac{0.967d}{1 + 0.967} + \frac{0.967d}{1 - 0.967}$$

$$35.88 \approx 29.79d. \qquad 2a \approx 35.88$$

So, $d \approx 1.204$ and

$$ed \approx (0.967)(1.204) \approx 1.164.$$

Using this value in the equation produces

$$r = \frac{1.164}{1 + 0.967 \sin \theta}$$

where r is measured in astronomical units. To find the closest point to the sun (the focus), you can write

$$c = ea \approx (0.967)(17.94) \approx 17.35.$$

Because c is the distance between the focus and the center, the closest point is

$$a - c \approx 17.94 - 17.35$$
$$\approx 0.59 \text{ AU}$$
$$\approx 55{,}000{,}000 \text{ miles.}$$

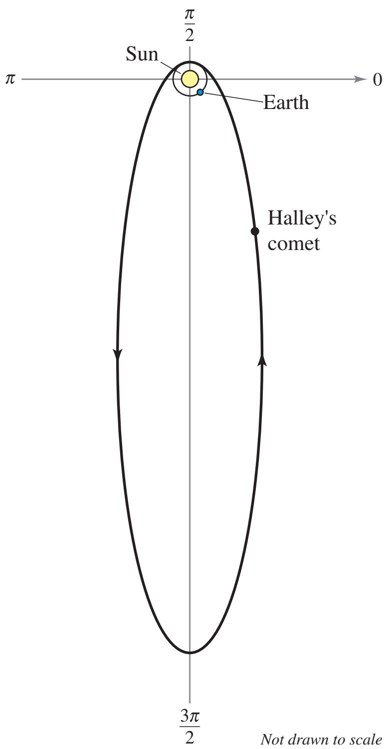

Figure 10.62

* If Earth is used as a reference with a period of 1 year and a distance of 1 astronomical unit, then the proportionality constant is 1. For example, because Mars has a mean distance to the sun of $D \approx 1.524$ AU, its period P is $D^3 = P^2$. So, the period for Mars is $P \approx 1.88$.

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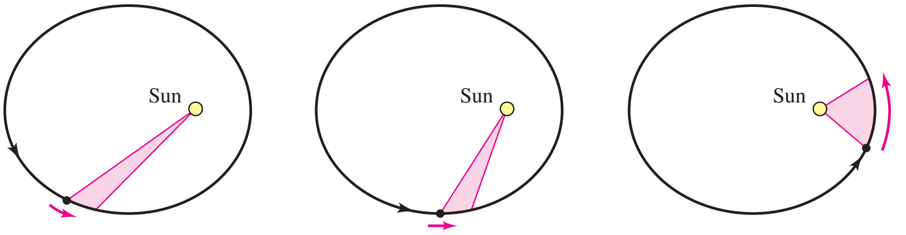
Kepler's Second Law states that as a planet moves about the sun, a ray from the sun to the planet sweeps out equal areas in equal times. This law can also be applied to comets or asteroids with elliptical orbits. For example, Figure 10.63 shows the orbit of the asteroid Apollo about the sun. Applying Kepler's Second Law to this asteroid, you know that the closer it is to the sun, the greater its velocity, because a short ray must be moving quickly to sweep out as much area as a long ray.



A ray from the sun to the asteroid Apollo sweeps out equal areas in equal times.
Figure 10.63

EXAMPLE 4 The Asteroid Apollo

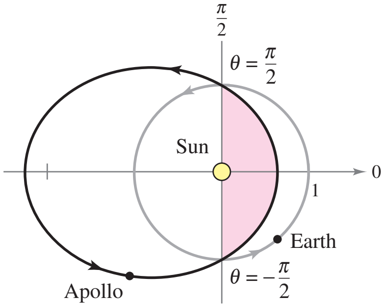


Figure 10.64

The asteroid Apollo has a period of 661 Earth days, and its orbit is approximated by the ellipse

$$r = \frac{1}{1 + (5/9)\cos\theta} = \frac{9}{9 + 5\cos\theta}$$

where r is measured in astronomical units. How long does it take Apollo to move from the position $\theta = -\pi/2$ to $\theta = \pi/2$, as shown in Figure 10.64?

Solution Begin by finding the area swept out as θ increases from $-\pi/2$ to $\pi/2$.

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta \qquad \text{Formula for area of a polar graph}$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(\frac{9}{9 + 5\cos\theta}\right)^2 d\theta$$

Using the substitution $u = \tan(\theta/2)$, as discussed in Section 8.6, you obtain

$$A = \frac{81}{112}\left[\frac{-5\sin\theta}{9 + 5\cos\theta} + \frac{18}{\sqrt{56}}\arctan\frac{\sqrt{56}\tan(\theta/2)}{14}\right]_{-\pi/2}^{\pi/2} \approx 0.90429.$$

Because the major axis of the ellipse has length $2a = 81/28$ and the eccentricity is $e = 5/9$, you can determine that

$$b = a\sqrt{1 - e^2} = \frac{9}{\sqrt{56}}.$$

So, the area of the ellipse is

$$\text{Area of ellipse} = \pi ab = \pi\left(\frac{81}{56}\right)\left(\frac{9}{\sqrt{56}}\right) \approx 5.46507.$$

Because the time required to complete the orbit is 661 days, you can apply Kepler's Second Law to conclude that the time t required to move from the position $\theta = -\pi/2$ to $\theta = \pi/2$ is

$$\frac{t}{661} = \frac{\text{area of elliptical segment}}{\text{area of ellipse}} \approx \frac{0.90429}{5.46507}$$

which implies that $t \approx 109$ days.

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10.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Graphical Reasoning **In Exercises 1–4, use a graphing utility to graph the polar equation when (a) $e = 1$, (b) $e = 0.5$, and (c) $e = 1.5$. Identify the conic.**

1. $r = \dfrac{2e}{1 + e\cos\theta}$

2. $r = \dfrac{2e}{1 - e\cos\theta}$

3. $r = \dfrac{2e}{1 - e\sin\theta}$

4. $r = \dfrac{2e}{1 + e\sin\theta}$

5. **Writing** Consider the polar equation

$$r = \frac{4}{1 + e\sin\theta}.$$

(a) Use a graphing utility to graph the equation for $e = 0.1$, $e = 0.25$, $e = 0.5$, $e = 0.75$, and $e = 0.9$. Identify the conic and discuss the change in its shape as $e \to 1^-$ and $e \to 0^+$.

(b) Use a graphing utility to graph the equation for $e = 1$. Identify the conic.

(c) Use a graphing utility to graph the equation for $e = 1.1$, $e = 1.5$, and $e = 2$. Identify the conic and discuss the change in its shape as $e \to 1^+$ and $e \to \infty$.

6. **Writing** Consider the polar equation

$$r = \frac{4}{1 - 0.4\cos\theta}.$$

(a) Identify the conic without graphing the equation.

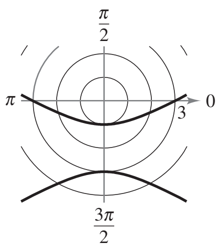
(b) Without graphing the following polar equations, describe how each differs from the polar equation above.

$$r = \frac{4}{1 + 0.4\cos\theta}, \quad r = \frac{4}{1 - 0.4\sin\theta}$$

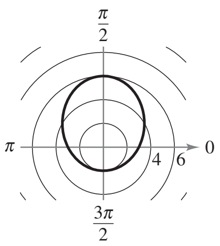
(c) Verify the results of part (b) graphically.

Matching **In Exercises 7–12, match the polar equation with the correct graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

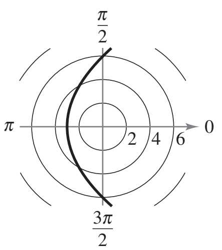
(a)



(b)



(c)



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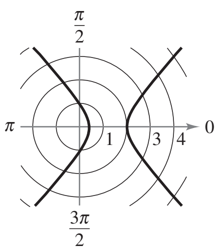

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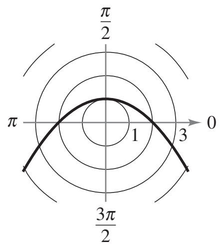
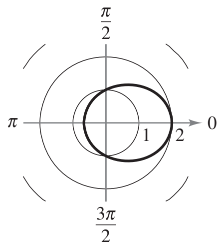


(f)



7. $r = \dfrac{6}{1 - \cos\theta}$

8. $r = \dfrac{2}{2 - \cos\theta}$

9. $r = \dfrac{3}{1 - 2\sin\theta}$

10. $r = \dfrac{2}{1 + \sin\theta}$

11. $r = \dfrac{6}{2 - \sin\theta}$

12. $r = \dfrac{2}{2 + 3\cos\theta}$

Sketching and Identifying a Conic **In Exercises 13–22, find the eccentricity and the distance from the pole to the directrix of the conic. Then sketch and identify the graph. Use a graphing utility to confirm your results.**

13. $r = \dfrac{1}{1 - \cos\theta}$

14. $r = \dfrac{6}{3 - 2\cos\theta}$

15. $r = \dfrac{3}{2 + 6\sin\theta}$

16. $r = \dfrac{4}{1 + \cos\theta}$

17. $r = \dfrac{5}{-1 + 2\cos\theta}$

18. $r = \dfrac{10}{5 + 4\sin\theta}$

19. $r = \dfrac{6}{2 + \cos\theta}$

20. $r = \dfrac{-6}{3 + 7\sin\theta}$

21. $r = \dfrac{300}{-12 + 6\sin\theta}$

22. $r = \dfrac{1}{1 + \sin\theta}$

Identifying a Conic **In Exercises 23–26, use a graphing utility to graph the polar equation. Identify the graph and find its eccentricity.**

23. $r = \dfrac{3}{-4 + 2\sin\theta}$

24. $r = \dfrac{-15}{2 + 8\sin\theta}$

25. $r = \dfrac{-10}{1 - \cos\theta}$

26. $r = \dfrac{6}{6 + 7\cos\theta}$

Comparing Graphs **In Exercises 27–30, use a graphing utility to graph the conic. Describe how the graph differs from the graph in the indicated exercise.**

27. $r = \dfrac{4}{1 + \cos(\theta - \pi/3)}$ (See Exercise 16.)

28. $r = \dfrac{10}{5 + 4\sin(\theta - \pi/4)}$ (See Exercise 18.)

29. $r = \dfrac{6}{2 + \cos(\theta + \pi/6)}$ (See Exercise 19.)

30. $r = \dfrac{-6}{3 + 7\sin(\theta + 2\pi/3)}$ (See Exercise 20.)

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31. Rotated Ellipse Write the equation for the ellipse rotated $\pi/6$ radian clockwise from the ellipse

$$r = \frac{8}{8 + 5\cos\theta}.$$

32. Rotated Parabola Write the equation for the parabola rotated $\pi/4$ radian counterclockwise from the parabola

$$r = \frac{9}{1 + \sin\theta}.$$

Finding a Polar Equation **In Exercises 33–44, find a polar equation for the conic with its focus at the pole. (For convenience, the equation for the directrix is given in rectangular form.)**

	Conic	**Eccentricity**	**Directrix**
33.	Parabola	$e = 1$	$x = -3$
34.	Parabola	$e = 1$	$y = 4$
35.	Ellipse	$e = \frac{1}{2}$	$y = 1$
36.	Ellipse	$e = \frac{3}{4}$	$y = -2$
37.	Hyperbola	$e = 2$	$x = 1$
38.	Hyperbola	$e = \frac{3}{2}$	$x = -1$

	Conic	**Vertex or Vertices**
39.	Parabola	$\left(1, -\frac{\pi}{2}\right)$
40.	Parabola	$(5, \pi)$
41.	Ellipse	$(2, 0),\ (8, \pi)$
42.	Ellipse	$\left(2, \frac{\pi}{2}\right),\ \left(4, \frac{3\pi}{2}\right)$
43.	Hyperbola	$\left(1, \frac{3\pi}{2}\right),\ \left(9, \frac{3\pi}{2}\right)$
44.	Hyperbola	$(2, 0),\ (10, 0)$

45. Finding a Polar Equation Find a polar equation for the ellipse with focus $(0, 0)$, eccentricity $\frac{1}{2}$, and a directrix at $r = 4 \sec\theta$.

46. Finding a Polar Equation Find a polar equation for the hyperbola with focus $(0, 0)$, eccentricity 2, and a directrix at $r = -8 \csc\theta$.

WRITING ABOUT CONCEPTS

47. Eccentricity Classify the conics by their eccentricities.

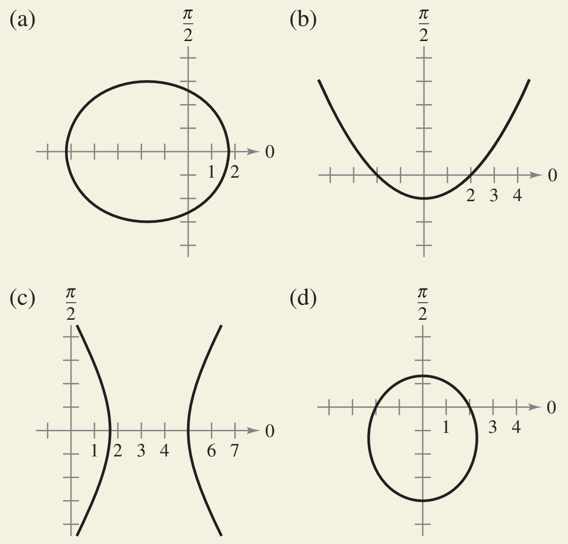
48. Identifying Conics Identify each conic.

(a) $r = \dfrac{5}{1 - 2\cos\theta}$ (b) $r = \dfrac{5}{10 - \sin\theta}$

(c) $r = \dfrac{5}{3 - 3\cos\theta}$ (d) $r = \dfrac{5}{1 - 3\sin(\theta - \pi/4)}$

49. Distance Describe what happens to the distance between the directrix and the center of an ellipse when the foci remain fixed and e approaches 0.

50. HOW DO YOU SEE IT? Identify the conic in the graph and give the possible values for the eccentricity.



51. Ellipse Show that the polar equation for $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$r^2 = \frac{b^2}{1 - e^2\cos^2\theta}.\qquad \text{Ellipse}$$

52. Hyperbola Show that the polar equation for $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$r^2 = \frac{-b^2}{1 - e^2\cos^2\theta}.\qquad \text{Hyperbola}$$

Finding a Polar Equation **In Exercises 53–56, use the results of Exercises 51 and 52 to write the polar form of the equation of the conic.**

53. Ellipse: focus at $(4, 0)$; vertices at $(5, 0)$, $(5, \pi)$

54. Hyperbola: focus at $(5, 0)$; vertices at $(4, 0)$, $(4, \pi)$

55. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

56. $\dfrac{x^2}{4} + y^2 = 1$

Area of a Region **In Exercises 57–60, use the integration capabilities of a graphing utility to approximate, to two decimal places, the area of the region bounded by the graph of the polar equation.**

57. $r = \dfrac{3}{2 - \cos\theta}$ **58.** $r = \dfrac{9}{4 + \cos\theta}$

59. $r = \dfrac{2}{3 - 2\sin\theta}$ **60.** $r = \dfrac{3}{6 + 5\sin\theta}$

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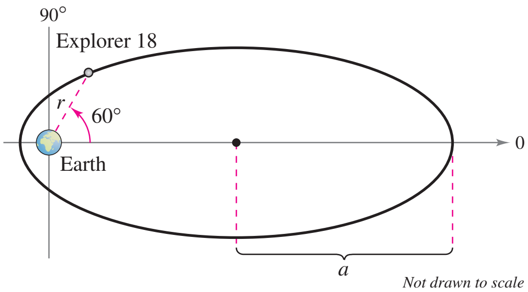
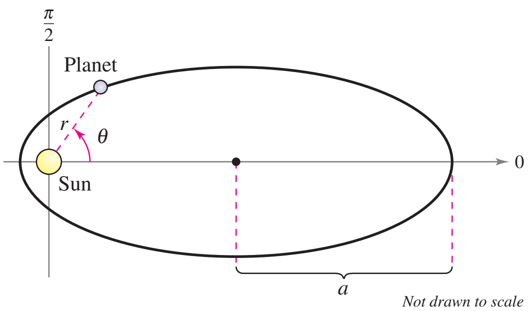
61. Explorer 18 On November 27, 1963, the United States launched Explorer 18. Its low and high points above the surface of Earth were approximately 119 miles and 123,000 miles (see figure). The center of Earth is a focus of the orbit. Find the polar equation for the orbit and find the distance between the surface of Earth and the satellite when $\theta = 60°$. (Assume that the radius of Earth is 4000 miles.)



62. Planetary Motion The planets travel in elliptical orbits with the sun as a focus, as shown in the figure.



(a) Show that the polar equation of the orbit is given by

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta}$$

where e is the eccentricity.

(b) Show that the minimum distance (*perihelion*) from the sun to the planet is $r = a(1 - e)$ and the maximum distance (*aphelion*) is $r = a(1 + e)$.

Planetary Motion In Exercises 63–66, use Exercise 62 to find the polar equation of the elliptical orbit of the planet, and the perihelion and aphelion distances.

63. Earth $a = 1.496 \times 10^8$ kilometers

$e = 0.0167$

64. Saturn $a = 1.427 \times 10^9$ kilometers

$e = 0.0542$

65. Neptune $a = 4.498 \times 10^9$ kilometers

$e = 0.0086$

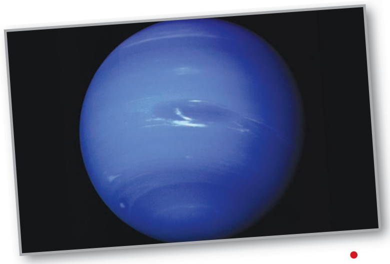
66. Mercury $a = 5.791 \times 10^7$ kilometers

$e = 0.2056$

NASA

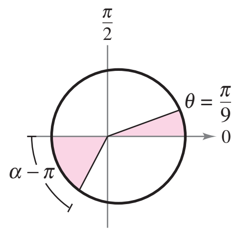
67. Planetary Motion

In Exercise 65, the polar equation for the elliptical orbit of Neptune was found. Use the equation and a computer algebra system to perform each of the following.



(a) Approximate the area swept out by a ray from the sun to the planet as θ increases from 0 to $\pi/9$. Use this result to determine the number of years required for the planet to move through this arc when the period of one revolution around the sun is 165 years.

(b) By trial and error, approximate the angle α such that the area swept out by a ray from the sun to the planet as θ increases from π to α equals the area found in part (a) (see figure). Does the ray sweep through a larger or smaller angle than in part (a) to generate the same area? Why is this the case?



(c) Approximate the distances the planet traveled in parts (a) and (b). Use these distances to approximate the average number of kilometers per year the planet traveled in the two cases.

68. Comet Hale-Bopp The comet Hale-Bopp has an elliptical orbit with the sun at one focus and has an eccentricity of $e \approx 0.995$. The length of the major axis of the orbit is approximately 500 astronomical units.

(a) Find the length of its minor axis.

(b) Find a polar equation for the orbit.

(c) Find the perihelion and aphelion distances.

Eccentricity In Exercises 69 and 70, let r_0 represent the distance from a focus to the nearest vertex, and let r_1 represent the distance from the focus to the farthest vertex.

69. Show that the eccentricity of an ellipse can be written as $e = \dfrac{r_1 - r_0}{r_1 + r_0}$. Then show that $\dfrac{r_1}{r_0} = \dfrac{1 + e}{1 - e}$.

70. Show that the eccentricity of a hyperbola can be written as $e = \dfrac{r_1 + r_0}{r_1 - r_0}$. Then show that $\dfrac{r_1}{r_0} = \dfrac{e + 1}{e - 1}$.

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